CHAPTER 9

Prime models and ω-categoricity

Type spaces provide a lot of information about a theory. In fact, certain model-theoretic properties of a theory turn out to correspond precisely with certain topological properties of the type spaces. We will see two examples of this phenomenon here: we will prove that a theory is ω-categorical if and only if all its type spaces are finite. We will also show that a theory has what is called a prime model if and only if the isolated points are dense in every type space of this theory. In proving these facts one uses quite heavily the properties of isolated and non-isolated types that we established in the previous chapter (that is, they rely on the fact that isolated types are realized in every model of a theory, while non-isolated types can be omitted).

I should add that what I wrote in the previous paragraph is true only for sufficiently nice theories. In fact, from now on we will often assume that a theory T

- is complete,
- has infinite models, and
- is formulated in a countable language.

If T satisfies these conditions, I will call T *nice* (this is not standard terminology). Note that nice theories have models of every infinite cardinality κ, do not have finite models and are such that every type over T is already realized in a countable model of T.

1. Atomic models

Before we embark on a study of ω-categoricity and prime models, we will first look at atomic models.

DEFINITION 9.1. A model A is *atomic* if it only realises isolated types in $S_n(\mathrm{Th}(A))$; put differently, a model is atomic if it omits all non-isolated types in $S_n(\mathrm{Th}(A))$.

Before we proceed, let us unwind this definition. Suppose A is an atomic model and $\overline{a}$ is a tuple of elements from A. Then, by definition, $p := \mathrm{tp}_A(\overline{a})$ is an isolated type over $\mathrm{Th}(A)$. This means that it contains a complete formula $\varphi(\overline{x})$ such that

$$\mathrm{Th}(A) \models \varphi(\overline{x}) \to \psi(\overline{x})$$

if and only if $\psi(\overline{x}) \in p$. What this does is reducing the "local question" whether $\overline{a}$ satisfies a formula $\psi(\overline{x})$ to the "global question" whether $A \models \varphi(\overline{x}) \to \psi(\overline{x})$. In other words, a model A is atomic if for any tuple $\overline{a}$ of elements from A there is formula $\varphi(\overline{x})$ such that for any formula $\psi(\overline{x})$ we have $A \models \psi(\overline{a})$ if and only if

$$A \models \varphi(\overline{x}) \to \psi(\overline{x}).$$

PROPOSITION 9.2. *If A is atomic and $\overline{a} \in A$, then $(A, \overline{a})$ is atomic as well.*

Proof. Let $\bar{b}$ be a tuple of elements from $(A, \overline{a})$. Look at $(\overline{a}, \bar{b})$. Since A is atomic there is a formula $\varphi(\overline{y}, \overline{x})$ with $A \models \varphi(\overline{a}, \bar{b})$ and

$$A \models \varphi(\overline{y}, \overline{x}) \to \psi(\overline{y}, \overline{z})$$

for every $\psi(\overline{y}, \overline{x})$ with $A \models \psi(\overline{a}, \bar{b})$. But then $\varphi(\overline{a}, \overline{x})$ is a formula satisfied by $\bar{b}$ such that

$$(A, \overline{a}) \models \varphi(\overline{a}, \overline{x}) \to \chi(\overline{x})$$

for every $\chi(\overline{x})$ with $(A, \overline{a}) \models \chi(\bar{b})$ (because each such $\chi(\overline{x})$ can be obtained from a formula $\psi(\overline{y}, \overline{x})$ with $\overline{a}$ substituted for $\overline{y}$). □

For the further study of atomic models we need the notion of an elementary map.

Definition 9.3. Let M and N be two L-structures. A partial function $f : X \subseteq M \to N$ from a subset X of M to N will be called an *elementary map* if

$$M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(f(m_1), \ldots, f(m_n))$$

for all L-formulas $\varphi(x_1, \ldots, x_n)$ and elements $m_1, \ldots, m_n \in X$. Note that this is equivalent to saying that $(M, x)_{x \in X} \equiv (N, fx)_{x \in X}$.

Proposition 9.4. *Let $f : \{a_1, \ldots, a_n\} \subseteq A \to M$ be an elementary map whose domain is a finite subset of an atomic model A. Then for any $a \in A$ there is an elementary map $g : \{a_1, \ldots, a_n\} \cup \{a\} \to M$ which extends f.*

Proof. Suppose $f : \{a_1, \ldots, a_n\} \subseteq A \to M$ is an elementary map whose domain is a finite subset of an atomic model A. Let us write $\overline{a}$ for the n-tuple $\langle a_1, \ldots, a_n \rangle$ and $f\overline{a}$ for the n-tuple $\langle fa_1, \ldots, fa_n \rangle$. The fact that f is an elementary map is equivalent to saying that $(A, \overline{a}) \equiv (M, f\overline{a})$.

So let $a \in A$. Since $(A, \overline{a})$ is atomic by the previous proposition, there is a formula $\varphi(x)$ such that $(A, \overline{a}) \models \varphi(a)$ and

$$(A, \overline{a}) \models \varphi(x) \to \psi(x)$$

for any formula $\psi(x)$ such that $(A, \overline{a}) \models \psi(a)$. Because $(A, \overline{a}) \models \varphi(a)$ and $(A, \overline{a}) \equiv (M, f\overline{a})$, we have $(A, \overline{a}) \models \exists x\, \varphi(x)$ and $(M, f\overline{a}) \models \exists x\, \varphi(x)$. So let $m \in M$ be such that $(M, f\overline{a}) \models \varphi(m)$. Then the type of a over $(A, \overline{a})$ and the type of m over $(M, f\overline{a})$ both contain the formula $\varphi(x)$, which is complete over $\mathrm{Th}(A, \overline{a}) = \mathrm{Th}(M, f\overline{a})$. This implies that these types are identical and we have $(M, \overline{a}, a) \equiv (M, f\overline{a}, m)$. So if we put $g(a) = m$ and $g(a_i) = f(a_i)$, then g is an elementary map extending f. □

Theorem 9.5. *Suppose A and M are two L-structures. If A is countable and atomic and $A \equiv M$, then A embeds elementarily into M.*

Proof. Suppose $\{a_0, a_1, a_2, \ldots\}$ is an enumeration of A. Using the previous proposition one can construct an increasing sequence of elementary maps $f_n : \{a_0, \ldots, a_n\} \to M$, starting with $f_0 = \emptyset$ (which is an elementary map as $A \equiv M$). But then $f = \bigcup_{n \in \mathbb{N}} f_n$ is an elementary embedding A into M. □

Theorem 9.6. *Suppose A and B are two L-structures which are both countable and atomic. If $A \equiv B$, then $A \cong B$.*

Proof. We use the back and forth method. So suppose $\{a_0, a_1, a_2, \ldots\}$ and $\{b_0, b_1, b_2, \ldots\}$ are enumerations of A and B, respectively. Using Proposition 9.4 one can construct an increasing sequence of elementary maps $f_n\colon X \subseteq A \to B$ such that $a_n \in \mathrm{dom}(f_{2n+1})$ and $b_n \in \mathrm{ran}(f_{2n+2})$, starting with $f_0 = \emptyset$. Then $f = \bigcup_{n\in\mathbb{N}} f_n$ is an isomorphism between A and B. □

2. Prime models

Prime models are closely connected to atomic models.

Definition 9.7. Let T be a theory. A model M of T is called *prime* if it can be elementarily embedded into any model of T.

Theorem 9.8. *A model of a nice theory T is prime iff it is countable and atomic.*

Proof. $\Rightarrow$: Let A be a prime model of a nice theory T. As a nice theory has countable models and A embeds in any model, A has to be countable as well. Moreover, if p is a non-isolated type of T, then there is a model B of T in which it is omitted, by the Omitting Types Theorem. Since A embeds elementarily into B, the type p will be omitted in A as well.

$\Leftarrow$: Let A be a countable and atomic model of a nice theory T and M be any other model of T. Since T is complete, we have $A \equiv M$, so A embeds elementarily into M by Theorem 9.5. □

Corollary 9.9. *Any two prime models of a nice theory T are isomorphic.*

Proof. This follows from Theorem 9.6 and Theorem 9.8. □

Theorem 9.10. *A nice theory T has a prime model iff the isolated n-types are dense in $S_n(T)$ for all n.*

Proof. Let us first translate the statement that that isolated n-types are dense in $S_n(T)$ in more logical terms. To say that the isolated types are dense means that every non-empty (basic) open set contains at least one isolated type: so any $[\varphi]$ which is not empty contains at least one isolated type p. But if p is isolated there is a complete formula ψ such that $\{p\} = [\psi] \subseteq [\varphi]$. So the isolated types are dense in $S_n(T)$ if every consistent formula $\varphi(\overline{x})$ is the consequence over T of some complete formula $\psi(\overline{x})$.

$\Rightarrow$: Let A be a prime model of T. Because a consistent formula $\varphi(\overline{x})$ is realised in *all* models of a complete theory, it is realized in A as well, by $\overline{a}$ say. Since A is atomic, $\varphi(\overline{x})$ belongs to the isolated type $\mathrm{tp}_A(\overline{a})$, so is the consequence over $\mathrm{Th}(A) = T$ of some complete formula $\psi(\overline{x})$.

$\Leftarrow$: Suppose isolated types are dense in every type space $S_n(T)$. Then we define for each natural number n a partial n-type

$$p_n(x_1, \ldots, x_n) = \{\, \neg\varphi(x_1, \ldots, x_n) : \varphi \text{ is complete} \,\},$$

and claim that these are not isolated. Because if p_n would be isolated there would be a consistent formula $\psi(\overline{x})$ such that

$$T \models \psi(\overline{x}) \to \neg\varphi(\overline{x})$$

for any complete formula $\varphi(\overline{x})$. But this would mean that $\psi(\overline{x})$ could not be a consequence of any complete formula, contradicting the fact that the isolated types are dense. So by the generalised omitting types theorem there is a countable model A omitting all p_n. But a structure omitting all p_n has to be atomic. □

3. ω-categoricity

We will now show that the results that we have proved can be used to draw various conclusions about ω-categorical theories.

THEOREM 9.11. (Ryll-Nardzewski Theorem) *For a nice theory T the following are equivalent:*

1. *T is ω-categorical;*
2. *all n-types are isolated;*
3. *all models of T are atomic;*
4. *all countable models of T are prime.*

PROOF. (1) $\Rightarrow$ (2): If $S_n(T)$ contains a non-isolated type p then there is a countable model where p is realized and a countable model where p is omitted (by the Omitting Types Theorem). So T cannot be ω-categorical.

(2) $\Rightarrow$ (3): If all types of a theory T are isolated, then any model of T can only realize isolated types. So all models of T are atomic.

(3) $\Rightarrow$ (4) follows from Theorem 9.8.

(4) $\Rightarrow$ (1) follows from Corollary 9.9. □

So a nice theory T is ω-categorical iff all types over T are isolated. But to say that every type is isolated means that there are only finitely many types.

PROPOSITION 9.12. *The following are equivalent for any theory T:*

1. *All n-types are isolated.*
2. *Every $S_n(T)$ is finite.*
3. *For for every n there are only finite many formulas $\varphi(x_1, \ldots, x_n)$ up to equivalence relative to T.*

PROOF. (1) $\Leftrightarrow$ (2) holds because $S_n(T)$ is a compact Hausdorff space.

(2) $\Rightarrow$ (3) If there are only finitely many n-types $p_1, \ldots, p_n$, then each p_i is isolated by some complete formula ψ_i. We claim that each formula with free variables among $x_1, \ldots, x_n$ is equivalent over T to some disjunction of the ψ_i, showing that up to logical equivalence there are only finitely many formulas with free variables among $x_1, \ldots, x_n$.

If φ is any formula with free variables among $x_1, \ldots, x_n$, then $[\varphi] \subseteq S_n(T)$, so

$$[\varphi] = \{p_i : i \in I\}$$

for some $I \subseteq \{1, \ldots, n\}$. But then

$$[\varphi] = \bigcup_{i \in I} \{p_i\} = \bigcup_{i \in I} [\psi_i] = [\bigvee_{i \in I} \psi_i],$$

so $T \models \varphi \leftrightarrow \bigvee_{i \in I} \psi_i$.

(3) $\Rightarrow$ (2): If every formula $\varphi(x_1, \dots, x_n)$ is equivalent modulo T to one of

$$\psi_1(x_1, \dots, x_n), \dots, \psi_m(x_1, \dots, x_n),$$

then every n-type is completely determined by saying which ψ_i it does and does not contain. □

Corollary 9.13. *If A is a model in a countable language and $\overline{a}$ is some tuple of elements from A, then $\mathrm{Th}(A)$ is ω-categorical iff $\mathrm{Th}(A, \overline{a})$ is ω-categorical.*

Proof. Every m-type $p(\overline{a}, \overline{x})$ of $\mathrm{Th}(A, \overline{a})$ determines an $(n+m)$-type $p(\overline{y}, \overline{x})$ of $\mathrm{Th}(A)$: so if there are only finitely many $(n+m)$-types consistent with $\mathrm{Th}(A)$, then there are only finitely many m-types consistent with $\mathrm{Th}(A, \overline{a})$.

Conversely, an m-type p consistent with $\mathrm{Th}(A)$ will be realized by some elements $\overline{c}$ in some elementary extension B of A. If $i \colon A \to B$ is the elementary embedding and $\overline{b} = i\overline{a}$, then $(B, \overline{b})$ is an elementary extension of $(A, \overline{a})$. Then $q = \mathrm{tp}_{(B, \overline{b})}(\overline{c})$ is a type over $\mathrm{Th}(A, \overline{a})$ extending p. Since $p \subseteq q$, these extensions q have to be different for different types p, and therefore the theory $\mathrm{Th}(A)$ cannot have more n-types than $\mathrm{Th}(A, \overline{a})$. So if the latter has only finitely many n-types, then so does the former. □

All of this has the following odd consequence. There are nice theories T_n having, up to isomorphism, n models, for $n = 1, 3, 4, 5, 6, \dots$ (see Exercise 1 below). But the case $n = 2$ is impossible.

Theorem 9.14. (Vaught's Theorem) *A nice theory cannot have exactly two countable models (up to isomorphism).*

Proof. If T is a nice theory which has more than one model (up to isomorphism), then T is not ω-categorical, so there must be some type p over T which is not isolated. But then there is a countable model A in which p is realized, by $\overline{a}$ say, and a model B in which p is omitted. Clearly, A and B cannot be isomorphic.

Since $T = \mathrm{Th}(A)$ is not ω-categorical, also $\mathrm{Th}(A, \overline{a})$ is not ω-categorical, by the previous corollary. So, again, there is a type q over $\mathrm{Th}(A, \overline{a})$ which is not isolated. Now we make a case distinction:

(1) If q is realized in $(A, \overline{a})$, let $(C, \overline{c})$ be a countable model in which it is omitted. Then C cannot be isomorphic to A; but C can also not be isomorphic to B because C realizes p, while B omits it.
(2) If q is omitted in $(A, \overline{a})$, let $(C, \overline{c})$ be a countable model in which it is realized. Then C cannot be isomorphic to A; but C can also not be isomorphic to B because C realizes p, while B omits it.

We conclude that any nice theory which is not ω-categorical must have at least three non-isomorphic models. □

4. Exercises

Exercise 1. Let $L_3 = \{<, c_0, c_1, c_2, \dots\}$, where $c_0, c_1, \dots$ are constant symbols. Let T_3 be the theory of dense linear orders with sentences added asserting $c_0 < c_1 < \dots$.

(a) Show that T_3 is a nice theory which has exactly three countable models up to isomorphism. *Hint:* Consider the questions: Does $c_0, c_1, c_2, \ldots$ have an upper bound? A least upper bound?

(b) Let $L_4 = L_3 \cup \{P\}$, where P is a unary predicate. Let T_4 be T_3 with the added sentences $P(c_i)$ and

$$\forall x\, \forall y\, \big(x < y \to \exists z\, \exists w\, (x < z < y \wedge x < w < y \wedge P(z) \wedge \neg P(w)) \big).$$

In other words, P is a dense-codense subset. Show that T_4 is a nice theory with exactly four countable models.

(d) Generalise (c) to give examples of nice theories which have exactly n countable models for $n = 5, 6, \ldots$

EXERCISE 2. Let T be the theory of $(\mathbb{R}, <, Q)$ where Q is a predicate for the rational numbers. Does T have a prime model?

EXERCISE 3. A theory T has *quantifier elimination* if for any formula $\varphi(\overline{x})$ there is a quantifier-free formula $\psi(\overline{x})$ such that

$$T \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x}).$$

(a) Suppose T is a nice ω-categorical theory and each $p \in S_n(T)$ contains a complete formula which is also quantifier-free. Deduce that T has quantifier elimination.

(b) Use (a) to show that $T = DLO$ and $T = RG$ have quantifier elimination.